

FOURIER-DOMAIN FIXED POINT ALGORITHMS WITH CODED DIFFRACTION PATTERNS

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ABSTRACT. Fourier-domain Difference Map (FDM) for phase retrieval with two oversampled coded diffraction patterns are proposed. FDM is a 3-parameter family of fixed point algorithms including Fourier-domain Hybrid-Projection-Reflection (FHPR) and Douglas-Rachford (FDR) algorithm. For generic complex objects without any object constraint, FDM yields a unique fixed point, after proper projection back to the object domain, that is the true solution to the phase retrieval problem up to a global phase factor.

1. INTRODUCTION

Fixed point algorithms are among the most effective algorithms for phase retrieval. These include Douglas-Rachford (DR) algorithm [1], Hybrid-Projection-Reflection (HPR) algorithm [2] and the Difference Map (DM) [4], all of which are based on the projections onto the constraint sets, including the object domain constraints (positivity, support constraint etc) and the Fourier magnitude constraint. Their performance is on a par with the industry standard such as the Hybrid-Input-Output (HIO) algorithm [8] which is not of the pure projection type and notoriously hard to analyze [1, 10].

The numerical challenge to any phasing algorithms is two-fold: the possibility of multiple fixed points and the non-convexity of the Fourier magnitude constraint. The latter is the nature of phase retrieval, independent of algorithms, while the former depends on the information content of the measured data as well as the design of algorithm.

The behaviors of any fixed point algorithm depend on the “landscape” of the object domain. If there are multiple attractive fixed points, the iterations can stagnate; if there are multiple hyperbolic fixed points, then a strange attractor may emerge and the iterations may exhibit a chaotic behavior. In other words, the presence of multiple fixed point in the object domain often severely deteriorate numerical performance, causing stagnation or even divergence of the iterations.

On the other hand, the presence of multiple fixed points in the Fourier domain may not be a bad thing, as long as these fixed points correspond to the unique fixed point in the object domain. On the contrary, the presence of multiple fixed points in the Fourier domain is a form of relaxation and may help mitigate the stagnation problem.

Therefore uniqueness of the fixed point in the object domain is a first-order concern to the algorithm design just like uniqueness of phase retrieval solution is to the measurement design. The latter, however, is the prerequisite of the former.

The purpose of the present work is to formulate the 3-parameter family of DM in the *Fourier domain with two oversampled coded diffraction patterns*, but without any object

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domain constraint, and prove the uniqueness of fixed point after proper projection back to the object domain.

The motivation for coded measurement is to the uniqueness of phase retrieval solution as established in [5] and the tremendous enhancement in numerical performance illustrated in [6, 7].

2. CODED DIFFRACTION PATTERNS

Let us first review the set-up for coded diffraction patterns.

Let $f(\mathbf{n})$ be a discrete object function with $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$. Consider the object space consisting of all functions supported in

$$\mathcal{N} = \{0 \leq n_1 \leq N_1, 0 \leq n_2 \leq N_2, \dots, 0 \leq n_d \leq N_d\}.$$

We assume $d \geq 2$.

With a coherent illumination under the Fraunhofer approximation, the free-space propagation between the object plane and the sensor plane can be described by the Fourier transform [3] (with the proper coordinates and normalization). However, only the *intensities* of the Fourier transform are measured on the sensor plane and constitute the so called *diffraction pattern* given by

$$\sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \sum_{\mathbf{m} \in \mathcal{N}} f(\mathbf{m} + \mathbf{n}) \overline{f(\mathbf{m})} e^{-i2\pi \mathbf{n} \cdot \boldsymbol{\omega}}, \quad \boldsymbol{\omega} = (w_1, \dots, w_d) \in [0, 1]^d, \quad \mathbf{N} = (N_1, \dots, N_d)$$

which is the Fourier transform of the autocorrelation

$$R_f(\mathbf{n}) = \sum_{\mathbf{m} \in \mathcal{N}} f(\mathbf{m} + \mathbf{n}) \overline{f(\mathbf{m})}.$$

Here and below the over-line notation means complex conjugacy.

Note that R_f is defined on the enlarged grid

$$\tilde{\mathcal{N}} = \{(n_1, \dots, n_d) \in \mathbb{Z}^d : -N_1 \leq n_1 \leq N_1, \dots, -N_d \leq n_d \leq N_d\}$$

whose cardinality is roughly 2^d times that of \mathcal{N} . Hence by sampling the diffraction pattern on the grid

$$\mathcal{L} = \left\{ (w_1, \dots, w_d) \mid w_j = 0, \frac{1}{2N_j + 1}, \frac{2}{2N_j + 1}, \dots, \frac{2N_j}{2N_j + 1} \right\}$$

we can recover the autocorrelation function by the inverse Fourier transform. This is the *standard oversampling* with which the diffraction pattern and the autocorrelation function become equivalent via the Fourier transform. The remaining task is to recover f from its autocorrelation function, the object domain constraints and the knowledge of μ .

A coded diffraction pattern is measured with a mask whose effect is multiplicative and results in a *masked object* of the form $f(\mathbf{n})\mu(\mathbf{n})$ where $\{\mu(\mathbf{n})\}$ is an array of random variables representing the mask. In other words, a coded diffraction pattern is just the plain diffraction pattern of a masked object.

We will focus on the effect of *random phases* $\phi(\mathbf{n})$ in the mask function $\mu(\mathbf{n}) = |\mu|(\mathbf{n})e^{i\phi(\mathbf{n})}$ where $\phi(\mathbf{n})$ are independent, continuous real-valued random variables and $|\mu|(\mathbf{n}) \neq 0, \forall \mathbf{n} \in \mathcal{L}$ (i.e. the mask is transparent).

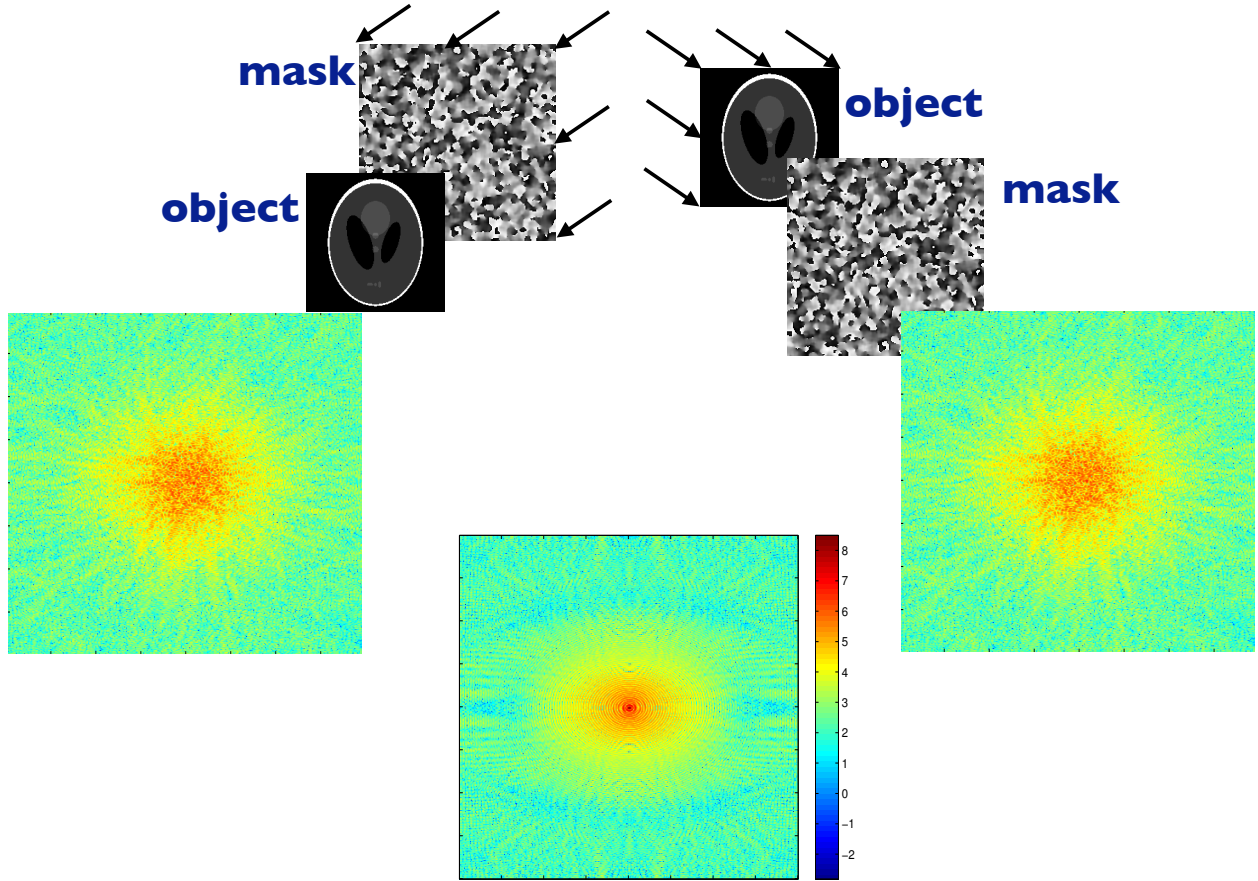


FIGURE 1. Conceptual layout of coherent lensless imaging with a random mask (left) *before* (for random illumination) or (right) *behind* (for wavefront sensing) the object (phantom). (middle) The diffraction pattern measured without a mask has a larger dynamic range. The color bar is on a logarithmic scale.

Accordingly, let Φ be the oversampled discrete Fourier transform from \mathcal{N} to \mathcal{L} such that $\Phi^* \Phi = I$. In other words, Φ is an isometry and has orthonormal columns.

In the case of one masked measurement, the measurement matrix is $\Psi = \Phi \text{diag}(\mu)$ whereas in the case of two masked measurements, the measurement matrix is given by

$$(1) \quad \Psi = \frac{1}{\sqrt{2}} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \quad \Psi_j = \Phi \text{diag}(\mu_j), \quad j = 1, 2.$$

where μ_1, μ_2 are two independently generated masks. Let $F = \Psi f$ be the mask-coded diffraction pattern(s).

Now we recall the uniqueness of phase retrieval solution with two coded diffraction patterns [5].

Proposition 1. [5] *Let f be a complex-valued object of dimension ≥ 2 . Let Ψ be the matrix given by (1). Let g be another complex object satisfying $|\Psi f| = |\Psi g|$ on \mathcal{L} . Then $g = e^{i\theta} f$, for some real constant θ , with probability one.*

3. DIFFERENCE MAP IN THE FOURIER DOMAIN

For ease of presentation, we shall assume that the masks are *phase masks*, i.e. $|\mu_1(\mathbf{n})| = |\mu_2(\mathbf{n})| = 1, \forall \mathbf{n}$. Consequently, $\Psi^* \Psi = I$.

For ease of notation, we convert the d -dimensional grid into an ordered set of index. For example, the unknown object $x_0 \in \mathbb{C}^{|\mathcal{N}|}$ is the vectorized version of the object function f originally supported in $\mathcal{N} \subset \mathbb{Z}^d, d \geq 2$.

Let $y \odot y'$ and y/y' be the component-wise multiplication and division between two vectors y, y' , respectively. For any $y \in \mathbb{C}^{|\mathcal{L}|}$ define the phase vector $\omega \in \mathbb{C}^{|\mathcal{L}|}$ with $\omega(j) = y(j)/|y(j)|$ where $|y(j)| \neq 0$. When $|y(j)| = 0$ the phase can be assigned arbitrarily and we set $\omega(j) = 1$ unless otherwise specified.

Phase retrieval can be formulated as the following feasibility problem in the Fourier domain

$$(2) \quad \text{Find } \hat{y} \in \Psi\mathcal{X} \cap \mathcal{Y}, \quad \mathcal{Y} := \{y \in \mathbb{C}^{|\mathcal{L}|} : |y| = b\}.$$

Let \mathcal{P}_o be the projection onto $\Psi\mathcal{X}$ and \mathcal{P}_m the projection onto \mathcal{Y} :

$$(3) \quad \mathcal{P}_o y = \Psi \Psi^* y, \quad \mathcal{P}_m y = b \odot \frac{y}{|y|}$$

The Difference Map (DM) \mathcal{D} is defined as follows. Let

$$(4) \quad \mathcal{D} = I + \beta \Delta$$

with

$$(5) \quad \Delta = \mathcal{P}_o((1 + \gamma_2)\mathcal{P}_m - \gamma_2 I) - \mathcal{P}_m((1 + \gamma_1)\mathcal{P}_o - \gamma_1 I)$$

where $\beta \neq 0, \gamma_1, \gamma_2$ are three relaxation parameters.

When $\gamma_1 = -1$ and $\gamma_2 = 1/\beta$,

$$(6) \quad \mathcal{D} = I + \beta \left(\mathcal{P}_o \left(\left(1 + \frac{1}{\beta}\right) \mathcal{P}_m - \frac{1}{\beta} I \right) - \mathcal{P}_m \right)$$

FDM becomes FHPR which, with $\beta = 1$, becomes FDR:

$$(7) \quad Sy = y + \Psi \Psi^* \left(2b \odot \frac{y}{|y|} - y \right) - b \odot \frac{y}{|y|}.$$

4. UNIQUENESS OF FIXED POINT

FDM is so designed that its Fourier fixed points become the phase retrieval solution after proper projection.

Let y_* be a fixed point of (4) and hence satisfy $\Delta y_* = 0$, i.e.

$$(8) \quad \mathcal{P}_o((1 + \gamma_2)\mathcal{P}_m - \gamma_2 I)y_* = \mathcal{P}_m((1 + \gamma_1)\mathcal{P}_o - \gamma_1 I)y_*.$$

Let

$$(9) \quad v_* \equiv ((1 + \gamma_1)\mathcal{P}_o - \gamma_1 I)y_*$$

$$(10) \quad \eta_* \equiv ((1 + \gamma_2)\mathcal{P}_m - \gamma_2 I)y_*$$

and

$$(11) \quad \hat{y} \equiv \mathcal{P}_o \eta_*, \quad \hat{x} \equiv \Psi^* \hat{y} = \Psi^* \eta_*.$$

By (8) $\hat{y} = \mathcal{P}_m v_* = \mathcal{P}_o \eta_*$ and hence \hat{y} satisfies both the object domain constraint represented by \mathcal{P}_o as well as the Fourier domain constraint represented by \mathcal{P}_m .

We now prove that DM produces the unique phase retrieval solution up to a constant phase factor.

Theorem 1. *Let y_* be a fixed point of FDM and $v_*, \eta_*, \hat{x}, \hat{y}$ be defined by (9), (10) and (11). Let $x_* = \Psi^* y_*$. The following statements hold with probability one.*

- (i) $\hat{y} = e^{i\theta} y_0$ and $\hat{x} = e^{i\theta} x_0$ for some real constant θ .
- (ii) If $\gamma_2 \neq 0$ and $\gamma_1 = -1$, then $\mathcal{P}_o y_* = e^{i\theta} y_0$ and $x_* = e^{i\theta} x_0$, for some real constant θ .
- (iii) If $\gamma_1 = 0$, then $y_* = e^{i\theta} y_0$ for some real constant θ .

Remark 1. *Part (i) means that in general \hat{y} , instead of y_* , is unique up to a constant phase factor. However, the relationship between \hat{y} and y_0 is nonlinear. For example, η_* and y_* are already related pixel-wise via the complicated relationship*

$$(12) \quad |\eta_*| = |(1 + \gamma_2)b - \gamma_2 y_*|$$

$$(13) \quad \angle \eta_* = \angle y_* + \sigma \pi$$

where σ can take any of the three values $0, \pm\pi$ depending on the pixel.

In view of part (ii), on the other hand, the relationship between y_* and y_0 is linear and the desirable property $x_* = e^{i\theta} x_0$ holds for FHPR with $\beta \neq 0$.

Part (iii) implies uniqueness in the Fourier domain (as well as in the object domain) up to a global phase factor.

Proof. Eq. (8) implies that $\mathcal{P}_o \eta_* = \mathcal{P}_m v_*$, namely $\mathcal{P}_o \eta_*$ shares the same *magnitude* as y_0 and the same *phase* as v_* at every point in \mathcal{L} :

$$(14) \quad |\mathcal{P}_o \eta_*| = |y_0|$$

$$(15) \quad \angle \mathcal{P}_o \eta_* = \angle v_*$$

on \mathcal{L} .

By Proposition 1 and (11), (14) implies

$$(16) \quad \hat{y} = e^{i\theta} y_0$$

for some real constant θ with probability one. This proves part (i).

For part (ii), substituting (16) into (10), we have

$$(17) \quad e^{i\theta} y_0 = (1 + \gamma_2) \mathcal{P}_o \mathcal{P}_m y_* - \gamma_2 \mathcal{P}_o y_*.$$

On the other hand, (15) implies that

$$(18) \quad \angle \hat{y} = \angle v_* = \angle y_*,$$

under the assumption $\gamma_1 = -1$, and hence $\mathcal{P}_m y_* = e^{i\theta} y_0$.

Now from (17) it follows that

$$\begin{aligned} e^{i\theta} y_0 &= e^{i\theta} (1 + \gamma_2) \mathcal{P}_o y_0 - \gamma_2 \mathcal{P}_o y_* \\ &= e^{i\theta} (1 + \gamma_2) y_0 - \gamma_2 \mathcal{P}_o y_* \end{aligned}$$

and, since $\gamma_2 \neq 0$,

$$(19) \quad \mathcal{P}_o y_* = e^{i\theta} y_0.$$

Applying Ψ^* on the both sides of (19), we obtain $x_* = e^{i\theta} x_0$.

For part (iii), we also need the uniqueness theorem of magnitude retrieval which requires only one coded diffraction pattern.

Proposition 2. [6, 9] *Let x_0 be a given rank ≥ 2 object. If*

$$(20) \quad \angle \Psi \hat{x} = \angle \Psi x_0$$

(after proper adjustment of the angles wherever the coded diffraction patterns vanish), then almost surely $\hat{x} = cx_0$ for some positive constant c .

With $\gamma_1 = 0$, $v_* = \mathcal{P}_o y_*$. By Proposition 2, (15) implies that $\Psi^* y_* = c \Psi^* \eta_*$ with a positive constant c . Hence from (16) we have $e^{i\theta} y_0 = v_*/c$.

Substituting $v_* = ce^{i\theta} y_0$ into (9) gives

$$(21) \quad \gamma_1 y_* = (1 + \gamma_1) \mathcal{P}_o y_* - ce^{i\theta} y_0.$$

Hence $y_* = \mathcal{P}_o y_*$ implying $y_* = ce^{i\theta} y_0$.

We claim that $c = 1$. This can be seen by substituting $y_* = ce^{i\theta} y_0$ into the fixed point equation (8) which becomes

$$(1 + \gamma_2) e^{i\theta} y_0 - \gamma_2 ce^{i\theta} y_0 = e^{i\theta} y_0$$

implying $c = 1$. □

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REFERENCES

- [1] H.H. Bauschke, P.L. Combettes and D. R. Luke, “Phase retrieval, error reduction algorithm, and Fienup variants: a view from convex optimization,” *J. Opt. Soc. Am. A* **19**, 13341-1345 (2002).
- [2] H.H. Bauschke, P.L. Combettes and D. R. Luke, “Hybrid projection-reflection method for phase retrieval,” *J. Opt. Soc. Am. A* **20**, 1025-1034 (2003).
- [3] M. Born and E. Wolf, *Principles of Optics*, 7-th edition, Cambridge University Press, 1999.
- [4] V. Elser, “Phase retrieval by iterated projections,” *J. Opt. Soc. Am. A* **20**, 40-55 (2003).
- [5] A. Fannjiang, “Absolute uniqueness of phase retrieval with random illumination,” *Inverse Problems* **28**, 075008 (2012).
- [6] A. Fannjiang and W. Liao, “Phase retrieval with random phase illumination”, *J. Opt. Soc. Am. A* **29**, 1847-1859 (2012).
- [7] A. Fannjiang and W. Liao, “Fourier phasing with phase-uncertain mask,” *Inverse Problems* **29**, 125001 (2013).
- [8] J.R. Fienup, “Phase retrieval algorithms: a comparison,” *Appl. Opt.* **21**, 2758-2769 (1982).
- [9] M. Hayes, “The reconstruction of a multidimensional sequence from the phase or magnitude of its Fourier Transform,” *IEEE Trans. Acoust. Speech and Sign. Proc.* **30**, 140-154 (1982).
- [10] S. Marchesini, “A unified evaluation of iterative projection algorithms for phase retrieval,” *Rev. Sci. Instr.* **78**, 011301 (2007).

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